RECOGNISING ACHIEVEMENT

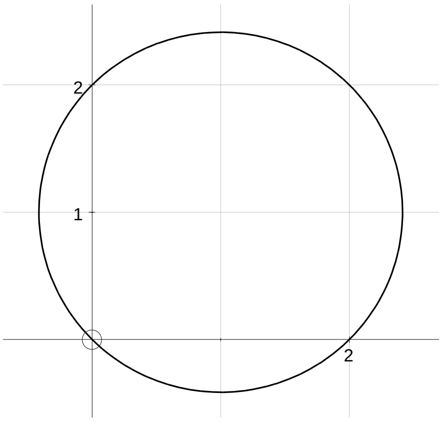
GCE

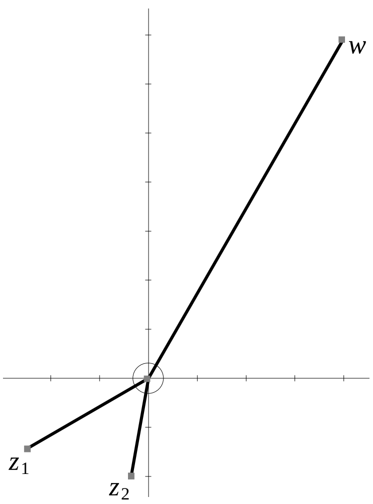
Mathematics (MEI)

Advanced GCE

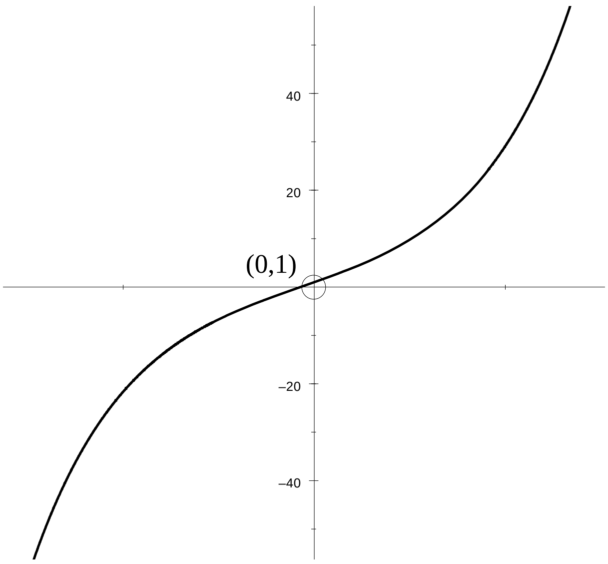
Unit **4756:** Further Methods for Advanced Mathematics

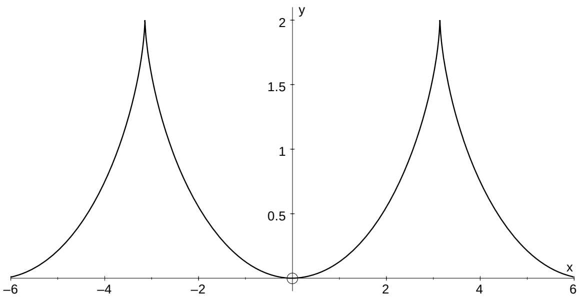
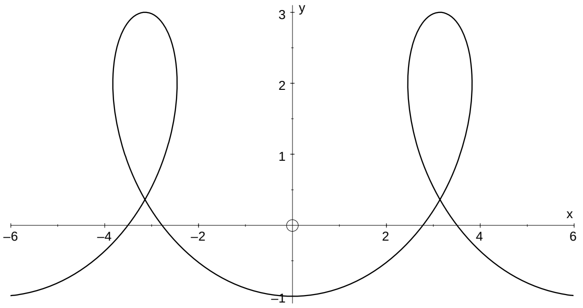
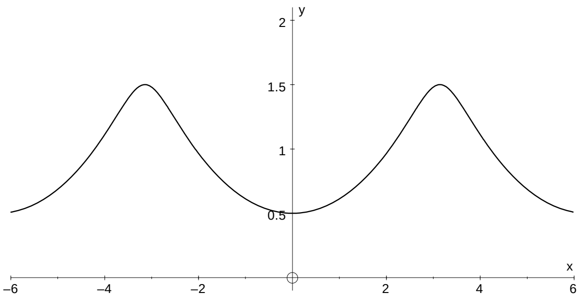
Mark Scheme for January 2011

1 (a)(i)	$x = r\cos\theta,\ y = r\sin\theta,\ x^2 + y^2 = r^2$ $r = 2(\cos\theta + \sin\theta)$ $\Rightarrow\ r^2 = 2r(\cos\theta + \sin\theta)$ $\Rightarrow\ x^2 + y^2 = 2x + 2y$ $\Rightarrow\ x^2 - 2x + y^2 - 2y = 0$ $\Rightarrow\ (x-1)^2 + (y-1)^2 = 2$ which is a circle centre (1, 1) radius $\sqrt{2}$  	M1 A1 (ag) M1 G1 G1 **5**	Using at least one of these Working must be convincing Recognise as circle or appropriate algebra leading to $(x-a)^2 + (y-b)^2 = r^2$ Attempt at complete circle with centre in first quadrant A circle with centre and radius indicated, or centre (1, 1) indicated and passing through (0, 0), or (2, 0) and (0, 2) indicated and passing through (0, 0)
(ii)	Area $= \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta$ $= 2\int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta)^2 \,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} (\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta)\,\mathrm{d}\theta$ $= 2\int_0^{\frac{\pi}{2}} (1 + 2\sin\theta\cos\theta)\,\mathrm{d}\theta$ $= 2\left[\theta - \frac{1}{2}\cos 2\theta\right]_0^{\frac{\pi}{2}}$ or $2\left[\theta + \sin^2\theta\right]_0^{\frac{\pi}{2}}$ etc. $= 2\left(\left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}\right)\right)$ $= \pi + 2$	 M1 M1 A1 A2 M1 A1 **7**	 Integral expression involving r^2 in terms of θ Multiplying out $\cos^2\theta + \sin^2\theta = 1$ used Correct result of integration with correct limits. Give A1 for one error Substituting limits. Dep. on both M1s Mark final answer
(b)(i)	$f'(x) = \frac{1}{2}\frac{1}{\left(1 + \frac{1}{4}x^2\right)} = \frac{2}{4 + x^2}$	M1 A1 **2**	Using Chain Rule Correct derivative in any form
(ii)	$f'(x) = \frac{1}{2}\left(1 + \frac{1}{4}x^2\right)^{-1} = \frac{1}{2}\left(1 - \frac{1}{4}x^2 + \frac{1}{16}x^4 - ...\right)$ $= \frac{1}{2} - \frac{1}{8}x^2 + \frac{1}{32}x^4 - ...$ $\Rightarrow f(x) = \frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{160}x^5 - ... + c$ But $c = 0$ because $\arctan(0) = 0$	M1 A1 M1 A1 A1 **5**	Correctly using binomial expansion Correct expansion Integrating at least two terms Independent **19**

2 (a)(i)	$z^n + z^{-n} = 2\cos n\theta$ $z^n - z^{-n} = 2\text{j}\sin n\theta$	B1 B1 **2**	
(ii)	$(z + z^{-1})^6 = z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6}$ $= z^6 + z^{-6} + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$ $\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $\Rightarrow \cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ $\Rightarrow \cos^6\theta = \frac{1}{32}\left(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right)$	M1 M1 A1 (ag) **3**	Expanding $(z + z^{-1})^6$ Using $z^n + z^{-n} = 2\cos n\theta$ with n = 2, 4 or 6. Allow M1 if 2 omitted, etc.
(iii)	$(z - z^{-1})^6 = z^6 + z^{-6} - 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) - 20$ $\Rightarrow -64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$ $\Rightarrow -\sin^6\theta = \frac{1}{32}\cos 6\theta - \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta - \frac{5}{16}$ $\Rightarrow \cos^6\theta - \sin^6\theta = \frac{1}{16}\cos 6\theta + \frac{15}{16}\cos 2\theta$	B1 M1 A1 M1 A1	Using (i) as in part (ii) Correct expression in any form Attempting to add or subtract
	OR $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$ B1 $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ M1 $\Rightarrow \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ A1 $\cos^6\theta - \sin^6\theta = 2\cos^6\theta - 3\cos^4\theta + 3\cos^2\theta - 1$ $\Rightarrow = \frac{1}{16}\cos 6\theta + \frac{15}{16}\cos 2\theta$ M1A1	**5**	This used Obtaining an expression for $\cos^4\theta$ Correct expression in any form Attempting to add or subtract
(b)(i)	$z_1{}^2 = 8e^{\frac{j\pi}{3}} \Rightarrow z_1 = 2\sqrt{2}e^{j\left(\frac{\pi}{6}+\pi\right)}$ $= 2\sqrt{2}e^{\frac{j7\pi}{6}}$ $z_2{}^3 = 8e^{\frac{j\pi}{3}} \Rightarrow z_2 = 2e^{j\left(\frac{\pi}{9}+\frac{4\pi}{3}\right)}$ $= 2e^{\frac{j13\pi}{9}}$ 	M1 A1 M1 A1 G1 G1 **6**	Correctly manipulating modulus and argument $\sqrt{8}$, $\frac{7\pi}{6}$ or $-\frac{5\pi}{6}$. Condone $r(c + js)$ Correctly manipulating modulus and argument 2, $\frac{13\pi}{9}$ or $-\frac{5\pi}{9}$. Condone $r(c + js)$ Moduli approximately correct Arguments approximately correct Give G1G0 for two points approximately correct
(ii)	$z_1 z_2 = 2\sqrt{2}e^{\frac{j7\pi}{6}} \times 2e^{\frac{j13\pi}{9}}$ $= 4\sqrt{2}e^{j\left(\frac{7\pi}{6}+\frac{13\pi}{9}\right)}$ $= 4\sqrt{2}e^{\frac{j11\pi}{18}}$ Lies in second quadrant	M1 A1 A1 **3**	Correctly manipulating modulus and argument Accept any equivalent form **19**

3 (i)	$\det(\mathbf{M} - \lambda\mathbf{I}) = (1-\lambda)[(3-\lambda)(1-\lambda)+8]$ $+ 4[2(1-\lambda) - 2] + 5[8 + (3-\lambda)]$ $= (1-\lambda)(\lambda^2 - 4\lambda + 11) + 4(-2\lambda) + 5(11-\lambda)$ $= -\lambda^3 + 5\lambda^2 - 15\lambda + 11 - 8\lambda + 55 - 5\lambda = 0$ $\Rightarrow \lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$	M1 A1 M1 A1 (ag) **4**	Obtaining det(**M** – λ**I**) Any correct form Simplification www, but condone omission of = 0
(ii)	$\lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$ $\Rightarrow (\lambda - 3)(\lambda^2 - 2\lambda + 22) = 0$ $\lambda^2 - 2\lambda + 22 = 0 \Rightarrow b^2 - 4ac = -84$ so no other real eigenvalues	M1 A1 M1 A1 **4**	Factorising and obtaining a quadratic. If M0, give B1 for substituting λ = 3 Correct quadratic Considering discriminant o.e. Conclusion from correct evidence www
(iii)	$\lambda = 3 \Rightarrow \begin{pmatrix} -2 & -4 & 5 \\ 2 & 0 & -2 \\ -1 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\Rightarrow -2x - 4y + 5z = 0$ $2x - 2z = 0$ $-x + 4y - 2z = 0$ $\Rightarrow x = z = k,\ y = \frac{3}{4}k$ $\Rightarrow$ eigenvector is $\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$ $\Rightarrow$ eigenvector with unit length is $\mathbf{v} = \frac{1}{\sqrt{41}}\begin{pmatrix} 4 \\ 3 \\ 4 \end{pmatrix}$ Magnitude of $\mathbf{M}^n\mathbf{v}$ is 3^n	 M1 M1 A1 B1 B1 **5**	 Two independent equations Obtaining a non-zero eigenvector Must be a magnitude
(iv)	$\lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$ $\Rightarrow \mathbf{M}^3 - 5\mathbf{M}^2 + 28\mathbf{M} - 66\mathbf{I} = \mathbf{0}$ $\Rightarrow \mathbf{M}^2 - 5\mathbf{M} + 28\mathbf{I} - 66\mathbf{M}^{-1} = \mathbf{0}$ $\Rightarrow \mathbf{M}^{-1} = \frac{1}{66}(\mathbf{M}^2 - 5\mathbf{M} + 28\mathbf{I})$	 M1 M1 A1 **3**	 Use of Cayley-Hamilton Theorem Multiplying by $\mathbf{M}^{-1}$ and rearranging Must contain **I** **16**

4 (i)	$\sinh t + 7\cosh t = 8$		
	$\Rightarrow \quad \frac{1}{2}(e^t - e^{-t}) + 7 \times \frac{1}{2}(e^t + e^{-t}) = 8$	M1	Substituting correct exponential forms
	$\Rightarrow \quad 4e^t + 3e^{-t} = 8$		
	$\Rightarrow \quad 4e^{2t} - 8e^t + 3 = 0$	M1	Obtaining quadratic in e^t
	$\Rightarrow \quad (2e^t - 1)(2e^t - 3) = 0$	M1	Solving to obtain at least one value of e^t
	$\Rightarrow \quad e^t = \frac{1}{2}$ or $\frac{3}{2}$	A1A1	Condone extra values
	$\Rightarrow \quad t = \ln(\frac{1}{2})$ or $\ln(\frac{3}{2})$	A1	These two values o.e. only. Exact form
		6	
(ii)	$\frac{dy}{dx} = 2\sinh 2x + 14\cosh 2x$ or $8e^{2x} + 6e^{-2x}$	B1	
	$2\sinh 2x + 14\cosh 2x = 16 \Rightarrow \sinh 2x + 7\cosh 2x = 8$		
	$\Rightarrow \quad 2x = \ln(\frac{1}{2})$ or $\ln(\frac{3}{2}) \Rightarrow x = \frac{1}{2}\ln(\frac{1}{2})$ or $\frac{1}{2}\ln(\frac{3}{2})$	M1 A1	Complete method to obtain an x value Both x co-ordinates in any exact form
	$x = \frac{1}{2}\ln(\frac{1}{2}) \Rightarrow y = -4 \qquad (\frac{1}{2}\ln(\frac{1}{2}), -4)$		
	$x = \frac{1}{2}\ln(\frac{3}{2}) \Rightarrow y = 4 \qquad (\frac{1}{2}\ln(\frac{3}{2}), 4)$	B1	Both y co-ordinates
	$\frac{dy}{dx} = 0 \Rightarrow 2\sinh 2x + 14\cosh 2x = 0$		
	$\Rightarrow \quad \tanh 2x = -7$ or $e^{4x} = -\frac{3}{4}$ etc.	M1	Any complete method
	No solutions because $-1 < \tanh 2x < 1$ or $e^x > 0$ etc.	A1 (ag)	www
		G1 G1	Curve (not st. line) with correct general shape (positive gradient throughout) Curve through (0, 1). Dependent on last G1
		8	
(iii)	$\int_0^a (\cosh 2x + 7\sinh 2x)\,dx = \frac{1}{2}$	M1	Attempting integration
	$\Rightarrow \quad \left[\frac{1}{2}\sinh 2x + \frac{7}{2}\cosh 2x\right]_0^a = \frac{1}{2}$	A1	Correct result of integration
	$\Rightarrow \quad \left(\frac{1}{2}\sinh 2a + \frac{7}{2}\cosh 2a\right) - \frac{7}{2} = \frac{1}{2}$		
	$\Rightarrow \quad \sinh 2a + 7\cosh 2a = 8$		
	$\Rightarrow \quad 2a = \ln(\frac{1}{2})$ or $\ln(\frac{3}{2}) \Rightarrow a = \frac{1}{2}\ln(\frac{1}{2})$ or $\frac{1}{2}\ln(\frac{3}{2})$	M1	Using both limits and a complete method to obtain a value of a
	$\Rightarrow \quad a = \frac{1}{2}\ln(\frac{3}{2}) \qquad (\frac{1}{2}\ln(\frac{1}{2}) < 0)$	A1	Must reject $\frac{1}{2}\ln(\frac{1}{2})$, but reason need not be given
		4	**18**

5 (i)	$a = 1$ 	G1	
	$a = 2$ 	G1	
	$a = 0.5$ 	G1	
		M2	Evidence s.o.i. of further investigation
(A)	Loops when $a > 1$	A1	
(B)	Cusps when $a = 1$	A1	
		7	
(ii)	If $x \to -x$, $t \to -t$ but $y(-t) = y(t)$ Curve is symmetrical in the y-axis	M1 A1 (ag) B1 **3**	Considering effect on t Effect on y
(iii)	$\dfrac{dy}{dx} = \dfrac{a\sin t}{1 + a\cos t}$	M1 A1	Using Chain Rule
	$\dfrac{dy}{dx} = 0 \Rightarrow a \sin t = 0 \Rightarrow t = 0$ and $\pm\pi$	A1	Values of t
	$t = 0 \Rightarrow$ T.P. is $(0, 1 - a)$ $t = \pm\pi \Rightarrow$ T.P. are $(\pm\pi, 1 + a)$	A1 A1 **5**	 Both, in any form
(iv)	$a = \frac{\pi}{2}$: both $t = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ give the point $(\pi, 1)$	B1 (ag)	Verification
	Gradients are a and $-a$ (or $\frac{\pi}{2}$ and $-\frac{\pi}{2}$)		
	Hence angle is $2\arctan(\frac{\pi}{2}) \approx 2.01$ radians	M1 A1 **3**	Complete method for angle Accept 115° (or 65°) **18**